Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 290 (2004) 316–323

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Oscillation criteria for a class of neutral difference equations with continuous variable

Shuhui Wu * and Zhanyuan Hou

Department of CCTM, London Metropolitan University, City Campus, 100 Minories, Tower Hill, London EC3N 1JY, UK

Received 19 May 2003

Submitted by R.P. Agarwal

Abstract

In this paper, we are mainly concerned with oscillatory behaviour of solutions for a class of second order nonlinear neutral difference equations with continuous variable. Using an integral transformation, the Riccati transformation and iteration, some oscillation criteria are obtained.

Keywords: Neutral difference equations; Oscillation; Continuous variable

1. Introduction

Recently, there has been an increasing interest in the study of oscillation of difference equations. Regarding the oscillatory behaviour of solutions, first order difference equations with continuous variable were studied in [1–5] and second order nonlinear difference equations, including neutral and advanced, were investigated in [6–10]. In this paper, we are mainly concerned with the second order nonlinear neutral difference equation

$$\Delta_\tau^2\big(x(t) - px(t-r)\big) + f\big(t, x\big(g(t)\big)\big) = 0, \tag{1.1}$$

where $p \geqslant 0$, τ and r are positive constants, $\Delta_\tau x(t) = x(t+\tau) - x(t)$, $0 < g(t) < t$, $g \in C^1([t_0,\infty), R^+)$ and $g'(t) > 0$, and $f \in C([t_0,\infty) \times R, R)$. Throughout this paper we assume that

* Corresponding author.
E-mail address: s.wu@londonmet.ac.uk (S. Wu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.09.065

$$g(t+\tau) \geqslant g(t)+\tau \quad \text{for } t \geqslant t_0 \tag{1.2}$$

and $f(t,u)/u \geqslant q(t) > 0$ for $u \neq 0$ and some $q \in C(R, R^+)$.

Let $t_0' = \min\{g(t_0), t_0 - r\}$ and $I_0 = [t_0', t_0]$. A function x is called the *solution* of (1.1) with $x(t) = \varphi(t)$ for $t \in I_0$ and $\varphi \in C(I_0, R)$ if it satisfies (1.1) for $t \geqslant t_0$.

A solution x is said to be *oscillatory* if it is neither eventually positive nor eventually negative; it is called *nonoscillatory* if it is not oscillatory.

We shall give our criteria in Section 2 and leave the proofs to Section 3.

2. Oscillation criteria

The assumptions given in Section 1 guarantee the existence and differentiability of the inverse g^{-1} of g. Let

$$\bar{q}(t) = \alpha \min_{t \leqslant s \leqslant t+2\tau}\bigl\{q(s)\bigr\}\Bigl(\min_{g(t) \leqslant s \leqslant g(t)+2\tau}\bigl\{\bigl(g^{-1}(s)\bigr)'\bigr\}\Bigr)^2, \tag{2.1}$$

where $0 < \alpha < 1$. We shall see below that oscillatory behaviour of the solutions of (1.1) can be determined by conditions involving the function $\bar{q}$. Let

$$z(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} x(\theta)\, d\theta,$$

where x denotes any solution of (1.1). Then $z''(t) = \Delta_\tau^2 x(t)$.

Theorem 1. *Assume that*

$$\sum_{i=0}^{\infty} \bar{q}(t' + i\tau) = \infty \tag{2.2}$$

for some $t' \geqslant t_0$. Then every solution of (1.1) *either is oscillatory or eventually satisfies $|z(t)| < p|z(t-r)|$.*

Remark. A special non-neutral case included in (1.1) is when $p = 0$. In this case, the condition (2.2) implies that every solution is oscillatory. In [8], the authors obtained oscillation criteria for a class of equations of the form

$$\Delta_\tau^2 x(t) + f\bigl(t, x(t-\sigma)\bigr) = 0.$$

We can see that even the special case of our Theorem 1 can be applied to a larger class of equations than the above.

Example 1. Consider the linear difference equation

$$\Delta_\tau^2\bigl(x(t) - px(t-r)\bigr) + \frac{1}{t}x\left(t - \frac{\sigma}{1+\beta t}\right) = 0 \tag{2.3}$$

for $t \geqslant 0$, where $p \geqslant 0$ and $\beta \geqslant 0$, r, τ and σ are positive constants. Viewing (2.3) as (1.1), we have $q(t) = 1/t$ and $g(t) = t - \sigma/(1+\beta t)$. Then, by (2.1), $\bar{q}(t) = \alpha/(t+2\tau)$ for $\beta = 0$ and

$$\bar{q}(t) = \frac{\alpha}{t+2\tau}\left(1 - \frac{\sigma\beta}{(1+\beta t)^2 + \sigma\beta}\right)^2$$

for $\beta > 0$. Since $\bar{q}(t) \geqslant \alpha'/(t+2\tau)$ for some $\alpha' > 0$ and all $t \geqslant 0$, $\bar{q}$ satisfied (2.2) with $t' = 0$. By Theorem 1, every solution of (2.3) either is oscillatory or eventually satisfies $|z(t)| < p|z(t-r)|$. In particular, when $p = 0$, every solution of (2.3) is oscillatory. It was shown in [8] that every solution of the equation $\Delta_\tau x(t) + t^{-1}x(t-\sigma) = 0$ is oscillatory. Clearly, this equation is a special case of (2.3) when $p = \beta = 0$.

Throughout this paper, we use the symbol $\lceil a \rceil$ to denote the smallest integer not less than a.

Theorem 2. *In addition to* (2.2), *we assume that* $0 < p < 1$ *and that there is a positive integer* k_0 *and a* $t_1 \geqslant t_0$ *satisfying* $m_n = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leqslant n$ *and*

$$\sum_{s=m_n}^{n} (s + 1 - m_n)\bar{q}(t_1 + s\tau) \geqslant \frac{(1-p)p^{k_0}}{1-p^{k_0}} \tag{2.4}$$

for large enough n. *Then every solution* x *of* (1.1) *is oscillatory.*

Theorem 3. *In addition to* (2.2), *we assume that* $p = 1$ *and that there is a positive integer* k_0 *and a* $t_1 \geqslant t_0$ *satisfying* $m_n = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leqslant n$ *and*

$$\sum_{s=m_n}^{n} (s + 1 - m_n)\bar{q}(t_1 + s\tau) \geqslant \frac{1}{k_0} \tag{2.5}$$

for large enough n. *Then every solution* x *of* (1.1) *is oscillatory.*

Theorem 4. *Under the conditions of Theorem* 2 *with the replacement of* $0 < p < 1$ *by* $p > 1$, *every bounded solution* x *of* (1.1) *is oscillatory.*

Example 2. Consider the difference equation

$$\Delta_\pi^2\big(x(t) - px(t-\pi)\big) + 8x(t-\pi) + \frac{8\sigma}{1+t^2}x^3(t-\pi) = 0, \tag{2.6}$$

where $\sigma \geqslant 0$ is a constant. Regarding (2.6) as (1.1), we have $\tau = \pi$, $r = \pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for any $\alpha \in (0, 1)$, $\bar{q} = 8\alpha$ by (2.1) so (2.2) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_n = n$ and

$$\sum_{s=m_n}^{n} (s + 1 - m_n)\bar{q}(t_1 + s\tau) = 8\alpha > 1 = \frac{1}{k_0}$$

if $\alpha > 1/8$. Moreover, we also have

$$\sum_{s=m_n}^{n} (s + 1 - m_n)\bar{q}(t_1 + s\tau) = 8\alpha > p = \frac{(1-p)p^{k_0}}{1-p^{k_0}}$$

if $p \in (0,1) \cup (1,8)$ and $\alpha > p/8$. By Theorems 1–4 every solution of (2.6) is oscillatory if $0 \leqslant p \leqslant 1$ and every bounded solution of (2.6) is oscillatory if $1 < p < 8$. If $p > 8$, then (2.6) with $\sigma = 0$ has a bounded positive solution $x(t) = \lambda^t$, where $y = \lambda^\pi$ is a root of $(y-p)(y-1)^2 + 8$ in $(0,1)$. Also, for $p \geqslant 1 + 3\sqrt[3]{2}$ and $\sigma = 0$, (2.6) has an unbounded positive solution $x(t) = \lambda^t$ for some $\lambda > 1$.

3. Proofs of the criteria

To prove the results given in Section 2, we need the following lemmas.

Lemma 1. *Assume that x is an eventually positive solution of* (1.1) *not satisfying $z(t) - pz(t-r) \to -\infty$ as $t \to \infty$. Then, with $u(t) = z(t) - pz(t-r)$, $u''(t) < 0$, $u'(t) > 0$, u is increasing and satisfies*

$$u''(t) + q(t)x\big(g(t)\big) \leqslant 0 \tag{3.1}$$

for t large enough.

Proof. From the assumption and (1.1) we have $x(g(t)) > 0$ and (3.1) for large enough t. So there is a $T \geqslant t_0$ such that $u''(t) < 0$ for $t \geqslant T$. We claim that $u'(t) > 0$ for $t \geqslant T$. Indeed, if not so, there is a $t_1 \geqslant T$ such that $u'(t_1) \leqslant 0$. Since $u''(t) < 0$ for $t \geqslant T$, we have $u'(t) \leqslant u'(t_1 + 1) < u'(t_1) \leqslant 0$ for $t \geqslant t_1 + 1$. This implies $u(t) \to -\infty$ as $t \to \infty$, which contradicts the assumption. Therefore, $u'(t) > 0$ for $t \geqslant T$ so that u is increasing. $\square$

Lemma 2. *Suppose x is an eventually positive solution of* (1.1) *not satisfying $z(t) - pz(t-r) \to -\infty$ as $t \to \infty$. Then $\Delta_\tau^2 u(t) < 0$, $\Delta_\tau u(t) > 0$, u is increasing and, for every integer $k \geqslant 0$, satisfies*

$$\Delta_\tau^2 u(t) + \bar{q}(t) \sum_{i=0}^{k} p^i u\big(g(t) - kr\big) \leqslant 0 \tag{3.2}$$

for sufficiently large t.

Proof. From Lemma 1 and the assumption, we have $\Delta_\tau^2 u(t) < 0$, $\Delta_\tau u(t) > 0$, $x(g(t)) > 0$ and (3.1) for some $T \geqslant t_0$ and all $t \geqslant T$. Then, for $t \geqslant T$, the assumptions on g and q give

$$\begin{aligned}
&\int_t^{t+\tau} ds \int_s^{s+\tau} x\big(g(\theta)\big)q(\theta)\,d\theta \\
&\quad \geqslant \min_{t \leqslant l \leqslant t+2\tau} \big\{q(l)\big\} \int_t^{t+\tau} ds \int_s^{s+\tau} x\big(g(\theta)\big)\,d\theta \\
&\quad \geqslant \min_{t \leqslant l \leqslant t+2\tau} \big\{q(l)\big\} \int_{g(t)}^{g(t+\tau)} \big(g^{-1}(s)\big)'\,ds \int_s^{g(g^{-1}(s)+\tau)} x(\theta)\big(g^{-1}(\theta)\big)'\,d\theta
\end{aligned}$$

$$\geqslant \min_{t\leqslant l\leqslant t+2\tau}\{q(l)\}\int_{g(t)}^{g(t)+\tau}\big(g^{-1}(s)\big)'\,ds\int_{s}^{s+\tau}x(\theta)\big(g^{-1}(\theta)\big)'\,d\theta$$
$$\geqslant \bar{q}(t)z\big(g(t)\big).$$

Hence, integrating (3.1), we have

$$\Delta_\tau^2 u(t)+\bar{q}(t)z\big(g(t)\big)\leqslant 0, \tag{3.3}$$

so

$$\Delta_\tau^2 u(t)+\bar{q}(t)\sum_{i=0}^{k-1}p^i u\big(g(t)-ir\big)+\bar{q}(t)p^k z\big(g(t)-kr\big)\leqslant 0$$

for $t\geqslant T$ and every integer $k>0$. Then (3.2) follows from this for large enough t as $\bar{q}(t)p^k z(g(t)-kr)>0$ and u is increasing. □

Proof of Theorem 1. Suppose the conclusion does not hold. Without loss of generality, let x be an eventually positive solution of (1.1) not eventually satisfying $z(t)<pz(t-r)$. By Lemmas 1 and 2, there is a $T\geqslant t_0$ such that $u''(t)<0$, $u'(t)>0$ and, for any positive integer k, (3.2) holds for $t\geqslant T$. As $u(t)<0$ is not eventually satisfied and u is increasing, we may assume $u(t)>0$ for $t\geqslant T$. Take $T_1>T$ such that $g(t)-kr\geqslant T$ for $t\geqslant T_1$. Note that $(\Delta_\tau u(t))'=\Delta_\tau u'(t)<0$ so $\Delta_\tau u(t)$ is decreasing. Hence, by (1.2),

$$u\big(g(t+\tau)-kr\big)-u\big(g(t)-kr\big)\geqslant \Delta_\tau u\big(g(t)-kr\big)>\Delta_\tau u(t)>0$$

for $t\geqslant T_1$. Then, by the Riccati transformation

$$v(t)=\frac{\Delta_\tau u(t)}{u(g(t)-kr)}, \tag{3.4}$$

we have $v(t)>0$ and

$$\Delta_\tau v(t)<\frac{\Delta_\tau^2 u(t)}{u(g(t)-kr)}-v(t)v(t+\tau)<0.$$

Thus, $v(t)>v(t+\tau)$. Further, from (3.2),

$$\Delta_\tau v(t)<-\bar{q}(t)\sum_{i=0}^{k}p^i-v^2(t+\tau)$$

so

$$\Delta_\tau v(t)+\bar{q}(t)\sum_{i=0}^{k}p^i+v^2(t+\tau)<0 \tag{3.5}$$

for $t\geqslant T_1$. There is an integer $N>0$ such that $t'+N\tau\geqslant T_1$. Now replacing t by $t'+j\tau$ and summing up both sides of (3.5) for j from N to n, we have

$$v\bigl(t' + (n+1)\tau\bigr) - v(t' + N\tau) + \sum_{j=N}^{n} \bar{q}(t' + j\tau) \sum_{i=0}^{k} p^i + \sum_{j=N}^{n} v^2\bigl(t' + (j+1)\tau\bigr) < 0.$$

Therefore, for all $n > N$,

$$\sum_{j=N}^{n} \bar{q}(t' + j\tau) \sum_{i=0}^{k} p^i < v(t' + N\tau) < \infty.$$

This contradicts (2.2) and, hence, shows the theorem. □

Proof of Theorem 2. Suppose the conclusion does not hold. Without loss of generality, let x be an eventually positive solution of (1.1). If $\lim_{t\to\infty} u(t) = -\infty$, then $u(t) = z(t) - pz(t-r) \leqslant 0$ for large enough t. Using this repeatedly and by the condition $0 < p < 1$, we obtain $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} u(t) = 0$. This contradiction shows that $u(t) \not\to -\infty$ as $t \to \infty$. Thus, the conclusions of Lemmas 1 and 2 hold. From (3.3), we have

$$\Delta_\tau^2 u(t) - \frac{\bar{q}(t)}{p} u\bigl(g(t) + r\bigr) + \frac{\bar{q}(t)}{p} z\bigl(g(t) + r\bigr) \leqslant 0.$$

Using the same technique as that used in the proof of Lemma 2, we obtain

$$\Delta_\tau^2 u(t) - \bar{q}(t) \sum_{i=1}^{k} \frac{1}{p^i} u\bigl(g(t) + i\tau\bigr) \leqslant 0. \tag{3.6}$$

As u is increasing and $\sum_{i=1}^{k} 1/p^i = (1-p^k)/[p^k(1-p)]$, (3.6) leads to

$$\Delta_\tau^2 u(t) \leqslant \bar{q}(t) \frac{1-p^k}{p^k(1-p)} u\bigl(g(t) + kr\bigr). \tag{3.7}$$

Replacing k by k_0 and t by $t_1 + i\tau$ in (3.7) and summing up both sides for i from s to n, we have

$$\Delta_\tau u\bigl(t_1 + (n+1)\tau\bigr) - \Delta_\tau u(t_1 + s\tau) \leqslant \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{i=s}^{n} \bar{q}(t_1 + i\tau) u\bigl(g(t_1 + i\tau) + k_0 r\bigr).$$

Then, summing up the above inequality for s from m_n to n, we obtain

$$\Delta_\tau u\bigl(t_1 + (n+1)\tau\bigr)(n + 1 - m_n) - u\bigl(t_1 + (n+1)\tau\bigr) + u(t_1 + m_n\tau) \leqslant \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{s=m_n}^{n} \sum_{i=s}^{n} \bar{q}(t_1 + i\tau) u\bigl(g(t_1 + i\tau) + k_0 r\bigr).$$

Combining this with $u(g(t_1 + i\tau) + k_0 r) \leqslant u(t_1 + m_n\tau)$ gives

$$\begin{aligned}
&\Delta_\tau u\big(t_1+(n+1)\tau\big)(n+1-m_n)-u\big(t_1+(n+1)\tau\big)\\
&\quad\leqslant u(t_1+m_n\tau)\left\{\frac{1-p^{k_0}}{p^{k_0}(1-p)}\sum_{s=m_n}^{n}(s+1-m_n)\bar q(t_1+s\tau)-1\right\}.
\end{aligned}$$

This inequality holds for large enough n as (3.7) holds for large enough t. By Theorem 1 and Lemma 2, $u(t)<0$ and $\Delta_\tau u(t)>0$ for large enough t. Hence, from the above inequality, we have

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar q(t_1+i\tau)<\frac{p^{k_0}(1-p)}{1-p^{k_0}}$$

for sufficiently large n. This contradiction to (2.4) shows that every solution of (1.1) is oscillatory. □

Proof of Theorem 3. The proof of Theorem 1 up to (3.7) is still valid when $z(t)\to 0$ and $u(t)\to 0$ as $t\to\infty$ is replaced by the boundedness of z and u due to $p=1$. With $p=1$, (3.6) and (3.7) now become

$$\Delta_\tau^2 u(t)-\bar q(t)\sum_{i=1}^{k_0}u\big(g(t)+ir\big)\leqslant 0 \tag{3.8}$$

and

$$\Delta_\tau^2 u(t)\leqslant k_0\bar q(t)u\big(g(t)+k_0r\big). \tag{3.9}$$

Replacing t by $t_1+i\tau$ in (3.9) and using the same technique as that in the proof of Theorem 2, we obtain

$$\begin{aligned}
&\Delta_\tau u\big(t_1+(n+1)\tau\big)(n+1-m_n)-u\big(t_1+(n+1)\tau\big)\\
&\quad\leqslant u(t_1+m_n\tau)\left\{k_0\sum_{s=m_n}^{n}(s-m_n+1)\bar q(t_1+i\tau)-1\right\}.
\end{aligned}$$

As $u(t)<0$ and $\Delta_\tau u(t)>0$ for large enough t, we must have

$$\sum_{s=m_n}^{n}(s-m_n+1)\bar q(t_1+i\tau)<\frac{1}{k_0}$$

for large enough n, which contradicts (2.5). Therefore, every solution of (1.1) is oscillatory. □

Proof of Theorem 4. Suppose (1.1) has a bounded eventually positive solution x so z is bounded. Then $\lim_{t\to\infty}u(t)\neq-\infty$. The rest is the same as the proof of Theorem 2. □

References

[1] I.P. Stavroulakis, Oscillation of delay difference equations, Comput. Math. Appl. 29 (1995) 83–88.

[2] Y. Zhang, J. Yan, Oscillation criteria for difference equations with continuous arguments, Acta Math. Sinica 38 (1995) 406–411.
[3] Y. Zhang, J. Yan, A. Zhao, Oscillation criteria for a difference equation, Indian J. Pure Appl. Math. 28 (1997) 1241–1249.
[4] B.G. Zhang, J. Yan, S.K. Choi, Oscillation for difference equations with continuous variable, Comput. Math. Appl. 36 (1998) 11–18.
[5] J. Yan, F. Zhang, Oscillation for system of delay difference equations, J. Math. Anal. Appl. 230 (1999) 223–231.
[6] Z.G. Zhang, J.F. Chen, C.S. Zhang, Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term, Comput. Math. Appl. 41 (2001) 1487–1494.
[7] Z.G. Zhang, J.L. Zhang, Oscillation criteria for second order advanced difference equations with "Summation Small" coefficient, Comput. Math. Appl. 38 (1999) 25–31.
[8] Z.G. Zhang, P. Bi, J.F. Chen, Oscillation of second order nonlinear difference equation with continuous variable, J. Math. Anal. Appl. 255 (2001) 349–357.
[9] R.P. Agarwal, M.M.S. Manuel, E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations II, Appl. Math. Lett. 10 (1997) 103–110.
[10] H.-J. Li, C.-C. Yeh, Oscillation criteria for second order neutral delay difference equations, Comput. Math. Appl. 36 (1998) 123–132.